ON DIRECTIONALLY PROXIMAL SUBDIFFERENTIAL

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Abstract. *In this paper, we study some properties of directionally proximal subdifferential and then we provide a necessary condition for directionally optimal solutions of the nonconstraint optimization problem.*

1 Introduction

For $f : \mathbb{R}^n \to \mathbb{R}$ is a function define at $\bar{x} \in \mathbb{R}^n$ and $\mathbb{R}^n \ni u \neq 0$, then f is said to be *differential in direction* u at $\bar{x}$ if the following limit is finite

$$\lim_{t \to 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t\|u\|}.$$

The above limit is called that *the derivative in direction* u at $\bar{x}$ of f and denoted by $f'_u(\bar{x})$.

For each $i = 1, \ldots, n$ consider $u_i := (\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0)$ then the derivative of f at $\bar{x}$ in direction u is called that the *partial derivative* of f with respect to the i^{th} variable.

The partial derivative is an important tool in the optimal theory. However, the class of the functions which exist the partial derivative is very exiguous. For example, the function $f(x, y) = \sqrt{|x| + |y|}$ does not exist the partial derivative at $\bar{x} = (0, 0)$ but it is easy to show that $\bar{x} = (0, 0)$ is the minimizer of f.

In 1960, Rokafellar [5] presented the subdifferential for the convex functions which is the generalization of the derivative. And then the subdifferential for the convex function was studied and obtained the important results.

Recently, Clark, Fréchet and Mordukhovich generalized the subdifferential for the convex functions become the Clark subdifferential, Fréchet subdifferential and limiting subdifferential for the nonconvex functions and stated many important results of the optimal theory due to these tools.

In 2012, Ginchev and Mordukhovich [2] presented the directionally subdifferential for the nonconvex functions and used this tool to establish the necessary condition for minimizer of the optimization problems. The directionally subdifferential is a (directionally) generalization of the partial derivative for the nonconvex and nondifferential functions.

In this paper, we present the notations on the directionally proximal normal cone and the directionally limiting normal cone. From these directionally normal cones, we establish the directionally proximal subdifferential and the directionally limiting subdifferential. Then we give some properties of these directionally subdifferentials

2 Preliminaries and auxiliary results

In this section, we always assume that X is a Hilbert space. For a sequence of subsets (A_k) of X, we present the *upper Painlewé-Kuratowski limits* and the *lower*

Painlewé-Kuratowski limits as follows

$$\operatorname*{Lim\,sup}_{k\to+\infty} A_k = \left\{x \in X \mid \exists k_m \to +\infty, \exists x_{k_m} \xrightarrow{m\to+\infty} x\right\}$$

and

$$\operatorname*{Lim\,inf}_{k\to+\infty} A_k = \left\{x \in X \mid \exists x_k \in A_k \text{ for large } k, \text{ with } x_k \xrightarrow{k\to+\infty} x\right\}.$$

The Painlewé-Kuratowski limit of the sequence (A_k), when the upper and lower limits coincide, is denoted by $\operatorname{Lim}_{k\to+\infty} A_k$.

In what follows, we will define the concepts on directionally normal cone and directionally subdifferential of a subset A of X. For $\bar{x} \in A, Q \subset X \setminus \{0\}$ and $\delta > 0$, one denotes by $D_Q(\bar{x};\delta) := B(\bar{x};\delta) \cap (\bar{x} + C_{Q_\delta})$ with $Q_\delta := Q + \delta B, C_Q := \{\lambda q \mid q \in Q, \lambda \geq 0\}$ and B is an unit ball. If $Q = \{u\}$ then we replace $C_{\{u\}}(\bar{x};\delta)$ by $C_u(\bar{x};\delta)$. Now we define *the proximal and limiting normal cones with respect to* Q as follows

$$N_Q^P(\bar{x};A) := \left\{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \leq p\|x - \bar{x}\|^2, \forall x \in A \cap D_Q(\bar{x};\delta) \text{ and some } p > 0\right\} \tag{2.1}$$

and

$$N_Q(\bar{x};A) := \operatorname*{Lim\,sup}_{x \xrightarrow{A,Q} \bar{x}} N_{Q_\delta}^P(x;A). \tag{2.2}$$

If $\bar{x} \notin A$ then we put $N_Q^P(\bar{x};A) := N_Q(\bar{x};A) := \emptyset$. If $Q = \{u\}$ then we call $N_u^P(\bar{x};A)$ and $N_u(\bar{x};A)$ respectively are *the proximal and limiting normal cones in direction* u.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ we denote the followings

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\} \text{ and } \operatorname{epi} f := \{(x,r) \mid r \geq f(x), x \in X\}.$$

Next, we define *the proximal and limiting subdifferential of f at $\bar{x}$ with respect to* Q as follows

$$\partial_Q^P f(\bar{x}) := \left\{x^* \in X^* \mid (x^*, -1) \in N_Q^P((\bar{x}, f(\bar{x})), \operatorname{epi} f)\right\}, \tag{2.3}$$

$$\partial_Q f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N_Q((\bar{x}, f(\bar{x})), \operatorname{epi} f)\}. \tag{2.4}$$

If $Q = \{u\}$ then we also denote by $\partial_u^P f(\bar{x})$ (resp. $\partial_u f(\bar{x})$), *the proximal subdifferential* (resp. *the limiting subdifferential*) *of f at $\bar{x}$ in direction u.*

A point $\bar{x} \in \operatorname{dom} f$ is called to be an *optimal minimizer of f in direction u* if there exists $\delta > 0$ such that $\bar{x} \in D_u(\bar{x};\delta)$ and $f(\bar{x}) \leq f(x)$ for all $x \in D_u(\bar{x};\delta)$.

3 Main results

We begin this section with the proposition on the convexity of directionally proximal normal cone and then we give the sum rules for directionally proximal subdifferential.

Proposition 3.1. *Let A be a subset of a Hilbert space X, $\bar{x} \in A$ and $u \in X \setminus \{0\}$. Then $N_u^P(\bar{x};A)$ is a convex normal cone.*

Proof. Let $x^*, y^* \in N_u^P(\bar{x}; A)$ and $z^* := \lambda x^* + (1-\lambda)y^*$ for some $\lambda \in [0;1]$ then one has

$$\langle x^*, x-\bar{x}\rangle \le \epsilon_1 \|x-\bar{x}\|^2 \text{ and } \langle y^*, x-\bar{x}\rangle \le \epsilon_2 \|x-\bar{x}\|^2 \text{ for every } x \in A \cap (\bar{x}+C_u).$$

So one gets

$$\langle z^*, x-\bar{x}\rangle = \lambda\langle x^*, x-\bar{x}\rangle + (1-\lambda)\langle y^*, x-\bar{x}\rangle \le (\lambda\epsilon_1 + (1-\lambda)\epsilon_2)\|x-\bar{x}\|^2$$

for every $x \in A \cap (\bar{x}+C_u)$. Setting $\epsilon := \max\{\epsilon_1; \epsilon_2\}$ then one obtains

$$\langle z^*, x-\bar{x}\rangle \le \epsilon\|x-\bar{x}\|^2 \text{ for every } x \in A \cap (\bar{x}+C_u)$$

which implies that $N_u^P(\bar{x}; A)$ is convex. □

Proposition 3.2. *Let $f : X \to \mathbb{R}$ is a lower semicontinuous function and $u \subset X \setminus \{0\}$ then one has*
(i) $\partial_u^P(cf)(\bar{x}) = c\partial_u^P f(\bar{x})$ *for all* $\bar{x} \in X$ *and* $c > 0$.
(ii) $\partial_u(cf)(\bar{x}) = c\partial_u f(\bar{x})$ *for all* $\bar{x} \in X$ *and* $c > 0$.

Proof. Let us first prove (i). Taking $x^* \in \partial_u^P(cf)(\bar{x})$ then one finds $\epsilon > 0, \delta > 0$ such that

$$\langle x^*, x-\bar{x}\rangle - (cf(x) - cf(\bar{x})) \le \epsilon\|x-\bar{x}\|^2 \text{ for all } x \in D_u(\bar{x},\delta) := B(\bar{x};\delta)\cap(\bar{x}+C_u).$$

It is equivalent to

$$\left\langle \frac{1}{c}x^*, x-\bar{x}\right\rangle - (f(x)-f(\bar{x})) \le \frac{\epsilon}{c}\|x-\bar{x}\|^2 \text{ for all } x \in D_u(\bar{x},\delta).$$

This implies that $\frac{x^*}{c} \in \partial_u^P f(\bar{x})$ and thus $x^* \in c\partial_u^P f(\bar{x})$.

Next we prove (ii). Taking $x^* \in \partial_u(cf)(\bar{x})$ then one finds sequences $\epsilon_n > 0, r_n \to 0, \delta_n \to 0, x_n \to \bar{x}$ and $x_n^* \xrightarrow{w^*} x^*$ such that

$$\langle x_n^*, x-x_n\rangle - (cf(x) - cf(x_n)) \le \epsilon_n\|x-x_n\|^2$$

for all $x \in D_u(x_n, r_n, \delta_n) := B(x_n; r_n)\cap(x_n + C_{u_{\delta_n}})$. It means that

$$\left\langle \frac{x_n^*}{c}, x-x_n\right\rangle - (f(x)-f(x_n)) \le \frac{\epsilon_n}{c}\|x-x_n\|^2 \text{ for all } x \in D_u(x_n,r_n,\delta_n).$$

This implies that $\frac{x^*}{c} \in \partial_u f(\bar{x})$ thus $x^* \in c\partial_u f(\bar{x})$ which complete the proof of theorem. □

Proposition 3.3. *Let $f : X \to \mathbb{R}$ be a lower semicontinuous function and $\bar{x}$ be a local solution in direction $u \in X \setminus \{0\}$ of f then $0 \in \partial_u^P f(\bar{x})$.*

Proof. Let $\bar{x}$ is a local solution in direction $u \in X \setminus \{0\}$ of f then there is $\delta > 0$ such that

$$f(x) - f(\bar{x}) \le 0 \text{ for all } x \in B(\bar{x},\delta)\cap(\bar{x}+C_{u_\delta}).$$

It implies that

$$\langle 0, x-\bar{x}\rangle - (f(x)-f(\bar{x})) \le 0 \le \epsilon\|x-\bar{x}\|^2 \text{ for every } \epsilon > 0.$$

Thus $0 \in \partial_u^P f(\bar{x})$. □

Finally, we complete this section with the fuzzy sum rule of the directionally proximal subdifferential. To obtain this, we let S be a subset of X and $\text{diam(S)} := \sup\{\|x-y\|,\ x,y \in S\}$. Let now us define the notation on the uniform lower semicontinuous in direction $u \in X$ of functions which is the generalization of the uniform lower semicontinuous in [3].

Definition 3.4. Let X be a Hilbert space and $f_j, j \in J$ be functions defined and lower semicontinuous in a neighborhood of $\bar{x}$and finite at $\bar{x}$ with J is the index finite set. We say that $f_j, j \in J$ are *uniform lower semicontinuous in direction* $u \in X\setminus\{0\}$ at $\bar{x}$ if there is $\delta > 0$ such that for any sequences $(x_{j,n}) \in D_u(\bar{x}, \delta)$ with $j \in J$ and such that $\lim_{n\to+\infty} \operatorname{diam}\{x_{j,n}, j \in J\} \to 0$ then there are $u_n \in D_u(\bar{x}, \delta)$ such that for all $j \in J, \lim_{n\to+\infty} \|x_{j,n} - u_n\| \to 0$ and

$$\liminf_{n\to+\infty} \sum_{j\in J} (f_j(x_{j,n} - f_j(u_n))) \geq 0.$$

Proposition 3.5. *Let X be a Hilbert space and $f_i, i \in J$ be (extended-real-valued) the uniform lower semicontinuous in direction $u \in X \setminus \{0\}$ at $\bar{x}$. Let $0 \in \partial_u^P(\sum_{i\in J} f_i)(\bar{x})$ then for any $\epsilon > 0$ there are $u_i, u_i^*, i \in J$ such that*

$$|f_i(u_i) - f_i(\bar{x})| \leq \epsilon; \qquad \|u_i - \bar{x}\| < \epsilon; \qquad u_i^* \in \partial_u^P f_i(\bar{u}_i); \qquad \|\sum_{i\in J} u_i^*\| < \epsilon.$$

Proof. It is without loss of generality that one can assume $f_i(\bar{x}) = 0$ for all $i \in J$. (If opposite then we replace $f_i(x)$ by $f_i(x) - f_i(\bar{x})$ for all $i \in J$.) Since $0 \in \partial_u^P(\sum_{i\in J} f_i)(\bar{x})$ by the definition, one finds $p > 0, \bar{\delta} > 0$ such that

$$\sum_{i\in J} f_i(x) \geq -p\|x - \bar{x}\|^2 \text{ with } x \in D_u(\bar{x}, \bar{\delta}). \tag{3.5}$$

Since $f_i, i \in J$ are uniform lower semicontinuous in direction u and $f_i(\bar{x}) = 0$ for all $i \in J$, one finds a $\hat{\delta} > 0$ which satisfying Definition 3.4 and $f_i(x) \geq -1$ for all $x \in D_u(\bar{x}; \hat{\delta})$ and $i \in J$. Putting $\delta := \min\left\{\bar{\delta}, \hat{\delta}\right\}$ and for each $n \in \mathbb{N}, x_i \in D_u(\bar{x}; \delta)$ with $i \in J$, one considers

$$f_n(x_i, i \in J) := \sum_{i\in J} f_i(x_i) + p\sum_{i\in J} \|x_i - \bar{x}\|^2 + n\sum_{i,j\in J} \|x_i - x_j\|^2.$$

Putting $\alpha_n := \inf\{f_n(x_i, i \in J) : x_i \in D_u(\bar{x}, \delta)\}$, one has

$$0 = f_n(\bar{x}, \ldots, \bar{x}) \geq \alpha_n \geq -|J| \text{ with } |J| \text{ is the card of } J.$$

For each $i \in J$, taking $x_{in} \in D_u(\bar{x}, \delta)$ to satisfy

$$f_n(x_{in}, i \in J) \leq \alpha_n + \frac{1}{n}.$$

Then one has

$$-|J| + n\sum_{i,j\in J} \|x_{in} - x_{jn}\|^2 \leq f_n(x_{in}, i \in J) \leq \frac{1}{n},$$

so that $\|x_{in} - x_{jn}\| \leq \sqrt{\sum_{i,j\in J} \|x_{in} - x_{jn}\|^2} \leq \sqrt{\frac{|J|}{n} + \frac{1}{n^2}}$ which converse 0 as n go to $+\infty$.

By the directionally uniform lower semicontinuity of $f_i, i \in J$ there are $u_n \in D_u(\bar{x}, \delta)$ such that $\|x_{jn} - u_n\| \to 0$ and

$$\sum_{j\in J} f_j(x_{jr}) \geq \sum_{j\in J} f_j(u_r) + o(1).$$

It follows that

$$0 \le \sum_{j\in J} f_j(u_n) + |J|p\|u_n - \bar{x}\|^2 \le \sum_{j\in J} f_j(x_{jn}) + p\sum_{j\in J}\|x_{jn} - \bar{x}\|^2 + o(1)$$
$$\le f_n(x_{in}, i \in J) + o(1) \le \frac{1}{n} + o(1), \qquad (3.6)$$

and $0 \le \Big|\|x_{jn} - \bar{x}\| - \|u_n - \bar{x}\|\Big| \le \|x_{jn} - u_n\| \to 0$. Hence $\sum_{j\in J} f_j(u_n) + |J|p\|u_n - \bar{x}\|^2 \to 0$ as $n \to +\infty$ and $u_n \to \bar{x}$ as well as all x_{jn}. It implies from (3.6) that

$$0 \le \sum_{j\in J} \liminf_{n\to+\infty} f_j(x_{jn}) \le \sum_{j\in J} \limsup_{n\to+\infty} f_j(x_{jn}) \le 0. \qquad (3.7)$$

One has $\lim_{n\to+\infty} f_j(x_{jn}) \ge \liminf_{x\to\bar{x}} f_j(x) \ge f_j(\bar{x}) = 0$. Combining to (3.7) one has $f_j(x_{jn}) \to 0$ for any $j \in J$.

For any small $\epsilon > 0$, it implies from $f_i(x_{in}) \to 0$ that there exists $n_0 \in \mathbb{N}$ such that $|f_i(x_{in})| < \frac{\epsilon}{2}$ for all $n \ge n_0$ and $i \in J$. Put $\delta := \frac{\epsilon}{2|J|}$, one finds a large number $r \in \mathbb{N}$ such that $x_{ir} \in \operatorname{cl} D_u(\bar{x};\delta) \subset D_u(\bar{x};\frac{\epsilon}{2})$ and

$$f_r(x_{ir}, i \in J) \le \inf_{D_u(\bar{z},\delta)} f_r + \delta. \qquad (3.8)$$

With $z = (x_i, i \in J) \in X$ one puts $D_u(z;\delta) := \prod_{i\in J} D_u(x_i;\delta),\ \bar{z} := (\bar{x},\ldots,\bar{x}), z^0 := (x_{ir}, i \in J) := (z_{i0}, i \in J) \in X^{|J|}$ and

$$T(z^0) := \left\{ z = (z_i, i \in J) \in \operatorname{cl} D_u(\bar{z},\delta) \mid f_r(z) + \frac{1}{2}\sum_{i\in J}\|z_i - z_{i0}\|^2 \le f_r(z^0) \right\}.$$

Then $z^0 \in T(z^0)$ and $T(z^0)$ is a nonempty closed set. Indeed, let us consider sequence $z^n := (z_{in}, i \in J) \subset T(z^0)$ with $\lim_{n\to+\infty} z^n := z := (z_i, i \in J)$ then $z \in \operatorname{cl} D(\bar{z};\delta)$ and one has

$$f_r(z^n) + \frac{1}{2}\sum_{i\in J}\|z_{in} - z_{i0}\|^2 \le f_r(z^0) \text{ for all } n = 0, 1, \ldots.$$

Since f_r and the norm function are lower semicontinuous, one has

$$f_r(z) + \frac{1}{2}\sum_{i\in J}\|z_i - z_{i0}\|^2 \le \liminf_{n\to+\infty}\left(f_r(z^n) + \frac{1}{2}\sum_{i\in J}\|z_{in} - z_{i0}\|^2 \right) \le f_r(z^0)$$

which implies that $z \in T(z^0)$.

For each $y = (y_i, i \in J) \in T(z^0)$ one has

$$\sum_{i\in J}\|y_i - z_{i0}\|^2 \le f_r(z^0) - f_r(y) \le f_r(z^0) - \inf_{D_u(\bar{z},\delta)} f_r \le \delta.$$

Taking $z^1 = (z_{i1}, i \in J) \in T(z^0)$ such that

$$f_r(z^1) + \sum_{i\in J}\|z_{i1} - z_{i0}\|^2 \le \inf_{z\in T(z^0)}\left\{ f_r(z) + \sum_{i\in J}\|z_i - z_{i0}\|^2 \right\} + \frac{\delta}{2}.$$

One again sets

$$T(z^1) := \left\{ z \in T(z^0) \mid f_r(z) + \sum_{i=0}^{1}\sum_{j\in J}\frac{1}{2^{i+1}}\|z_j - z_{ji}\|^2 \le f_r(z^1) + \frac{1}{2}\sum_{i\in J}\|z_{i1} - z_{i0}\|^2 \right\}.$$

Then $z^1 \in T(z^1)$ and $T(z^1)$ is a nonempty closed set.

In general, one defines

$$T(z^n) := \Big\{ z \in T(z^{n-1}) \mid f_r(z) + \sum_{i=0}^{n} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_j - z_{ji}\|^2 \le f_r(z^n) + \sum_{i=0}^{n-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_{ji} - z_{j0}\|^2 \Big\}$$

with $z^n = (z_{jn}, j \in J) \in T(z^{n-1})$ such that

$$f_r(z^n) + \sum_{i=0}^{n-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_{ji} - z_{j0}\|^2 \le \inf_{T(z^{n-1})} \left\{ f_r(z) + \sum_{i=0}^{n-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_j - z_{ji}\|^2 \right\} + \frac{\delta}{n2^n}.$$

Then $T(z^n)$ is also nonempty closed set.

For each $y := (y_i, \in J) \in T(z^n)$ one has

$$\frac{1}{2^{n+1}} \sum_{i\in J} \|y_i - z_{in}\|^2 \le \left[f_r(z^n) + \sum_{i=0}^{n-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_{j0} - z_{ji}\|^2 \right]$$

$$- \left[f_r(y) + \sum_{i=0}^{n-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_j - z_{ji}\|^2 \right] \le \inf_{T(z^{n-1})} \left\{ f_r(z) + \sum_{i=0}^{n-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_j - z_{ji}\|^2 \right\}$$

$$+ \frac{\delta}{n2^n} - \left[f_r(y) + \sum_{i=1}^{n-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|y_j - z_{ji}\|^2 \right] \le \frac{\delta}{n2^n}$$

and then one has

$$\sum_{i\in J} \|y_i - z_{in}\|^2 \le \frac{2\delta}{n} \tag{3.9}$$

which implies by Cauchy-Swchart inequality that $\sum_{i\in J} \|y_i - z_{in}\| \le \sqrt{\frac{\epsilon}{n}}$. Thus diameter of $T(z^n) \to 0$ as $n \to +\infty$ and hence (z^n) is a Cauchy sequence. It implies that there is $\hat{z} \in T(z^n)$ for $n = 1, 2, \ldots$ such that $z^n \to \hat{z} \in \operatorname{cl} D_u(\bar{z}; \delta)$. Finally for any $\operatorname{cl} D_u(\bar{z}, \delta) \ni z \neq \hat{z}$ one has $z \notin \bigcap_{i=0}^{+\infty} T(z^n)$ and so there is an $m \in \mathbb{N}$ such that

$$f_r(z) + \sum_{i=0}^{m} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_j - z_{ji}\|^2 > f_r(z^m) + \sum_{i=0}^{m-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_{jm} - z_{ji}\|^2. \tag{3.10}$$

Otherwise for any $q \ge m$ one has

$$f_r(z^m) + \sum_{i=0}^{m-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_{jm} - z_{ji}\|^2 \ge f_r(z^q) + \sum_{i=0}^{q-1} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_{jq} - z_{ji}\|^2$$

$$\ge f_r(\hat{z}) + \sum_{i=0}^{q} \sum_{j\in J} \frac{1}{2^{i+1}} \|\hat{z}_j - z_{ji}\|^2. \tag{3.11}$$

For each $j \in J$ one puts $z_j = \hat{z}_j + h_j$ with $h_j = \lambda_j u$ and puts $\lambda_\nu = 0$ if $\nu \neq j$ then one has

$$f_j(\hat{z}_j + h_j) - f_j(\hat{z}_j) + \sum_{i=m+1}^{q} \frac{1}{2^{i+1}} \|\hat{z}_j + h_j - z_{ji}\|^2 - \sum_{i=0}^{q} \frac{1}{2^{i+1}} \|\hat{z}_j - z_{ji}\|^2 \ge 0.$$

Thus one has

$$f_j(\hat{z}_j + h_j) - f_j(\hat{z}_j) + \|h_j\|^2 + 2\sum_{i=0}^{n} \frac{1}{2^{i+1}} \langle h_j, \hat{z}_j - z_{ji} \rangle \geq 0. \tag{3.12}$$

for all enough large n. Putting $u^*_{jn} := 2\sum_{i=0}^{n} \dfrac{1}{2^{i+1}}(\hat{z}_j - z_{ji})$ then we have

$$f_j(\hat{z}_j + h_j) - f_j(\hat{z}_j) + \|h_j\|^2 + \langle u^*_{jn}, h_j \rangle \geq 0. \tag{3.13}$$

Since sequence $\left(u^*_{jn}\right)$ is convergent so there is u^*_j such that $u^*_j := \lim_{n\to+\infty} u^*_{jn}$. Then for any $\hat{z}_j \neq x \in \operatorname{cl} D_u(\bar{x}, \delta)$ we have

$$f_j(x) - f_j(\hat{z}_j) + \|x - \hat{z}_j\|^2 + \langle u^*_j, x - \hat{z}_j \rangle \geq 0$$

which implies that $u^*_j \in \partial_u f_j(\hat{z}_j)$.

One now shows that $\sum_{j\in J} \|u^*_j\| < \epsilon$. Indeed, one has

$$\sum_{j\in J} \|u^*_{jn}\| = \sum_{j\in j} \|2\sum_{i=0}^{n} \frac{1}{2^{i+1}}(\hat{z}_j - z_{ji})\|$$

and noting that $z_n, \hat{z} \in \operatorname{cl} D_u(\bar{z}, \delta)$ which implies that $\|z_{jn} - \hat{z}_j\| < 2\delta = \frac{\epsilon}{|J|}$ for all $j \in J$ and $n = 0, 1, \ldots$. Thus one has $\sum_{j\in J} \|u^*_j\| < \epsilon$.

It remains to show that $|f_j(\hat{z}_j)| \leq \epsilon$ for all $j \in J$. Indeed, for each $j \in J$ and with large enough m it implies from (3.10) and (3.11) that

$$f_r(z) + \sum_{i=0}^{m} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_j - z_{ji}\|^2 \geq f_r(\hat{z}) + \sum_{i=0}^{q} \sum_{j\in J} \frac{1}{2^{i+1}} \|\hat{z}_j - z_{ji}\|^2$$

for any $z \in \operatorname{cl} D_u(\bar{z}; \delta)$ and $q \geq m$. Choosing $z = z^0$ in the above inequality and combining (3.9) one obtains

$$\inf_{\operatorname{cl} D_u(\bar{z};\delta)} f_r + 2\delta \geq f_r(z^0) + \delta \geq f_r(z^0) + \sum_{i=0}^{m} \sum_{j\in J} \frac{1}{2^{i+1}} \|z_{j0} - z_{ji}\|^2 \geq f_r(\hat{z}).$$

$$\frac{\epsilon}{|J|} \geq \sum_{i\in J} f_i(\hat{z}_i) + p\sum_{i\in J} \|\hat{z}_i - \bar{x}\|^2 + r\sum_{i,j\in J} \|\hat{z}_i - \hat{z}_j\|^2 \tag{3.14}$$

for any large enough $r \in \mathbb{N}$. It implies from (3.8) and (3.14) that

$$-\epsilon \leq -\frac{(|J|+1)\epsilon}{2|J|} \leq f_j(\hat{z}_j) \leq \frac{(|J|+1)\epsilon}{2|J|} \leq \epsilon.$$

$\square$

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