PAINLEVÉ-KURATOWSKI UPPER CONVERGENCE OF THE SOLUTION SETS FOR VECTOR QUASI-EQUILIBRIUM PROBLEMS

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Abstract. *In this paper, we study Painlevé-Kuratowski upper convergence of the solution sets with a sequence of mappings converging continuously for the vector quasi-equilibrium problems. Illustrating examples are provided.*

1 Introduction

The vector equilibrium problems contain many problems as special cases, including vector variational inequality problems, vector optimization problems, vector complementarity problems, vector Nash equilibrium problems, etc. Because of the general form of this problem, in fact it was investigated earlier under other terminologies. Recently, there has been an increasing interest in the study to stability of vector equilibrium problems.

As for the stabile result investigated on the convergence of the sequence of mappings, there are some results for the vector optimization, vector variational inequality problems and vector equilibrium problems with a sequence of sets converging in the sense of Painlevé-Kuratowski (see e.g., [4–6, 9, 11]). In [6], Huang discussed the convergence of the approximate efficient sets to the efficient sets of vector-valued and set-valued optimization problems in the sense of Painlevé-Kuratowski and Mosco. In [5], Fang et al. investigated the Painlevé-Kuratowski set convergence of the solution sets of the perturbed set-valued weak vector variational inequalities. In [9], Lalitha and Chatterjee investigated the Painlevé-Kuratowski set convergence of the solution sets of the a nonconvex vector optimization problem. In [11], Peng and Yang investigated the Painlevé-Kuratowski set convergence of the solution sets of the perturbed vector equilibrium problems without monotonicity in real linear metric spaces.

In this paper, we establish Painlevé-Kuratowski upper convergence of the solution sets of the vector quasi-equilibrium problems (in short,$(\text{QEP})_n$) with a sequence of converging mappings in normed vector spaces.

The structure of our paper is as follows. In Section 2, we introduce the problems (QEP) and $(\text{QEP})_n$, recall some definitions and important properties. In Section 3, we investigate Painlevé-Kuratowski upper convergence of the solution sets. Some examples are given for the illustration of our results.

2 Preliminaries

Throughout this paper, let X, Y be two normed vector spaces, A be a nonempty, compact and convex subset of X. Let $K : A \to 2^A$ be a set-valued mapping and $\varphi : A \times A \to Y$ be a vector function. C is a proper, closed and convex cone of Y, with $\text{int}C \neq \emptyset$.

We consider the vector quasiequilibrium problem.

(QEP) Finding $\bar{x} \in K(\bar{x})$ such that

$$\varphi(\bar{x}, y) \subseteq Y \setminus -\text{int}C, \ \forall y \in K(\bar{x}).$$

For sequences of set-valued mappings $K_n : A \to 2^A$ and sequences of functions $\varphi_n : A \times A \to Y$, we consider the sequence of the vector quasiequilibrium problems.

(QEP)$_n$ Finding $\bar{x}_n \in K_n(\bar{x}_n)$ such that

$$\varphi_n(\bar{x}_n, y) \subseteq Y \setminus -\text{int}C, \ \forall y \in K_n(\bar{x}_n).$$

We denote the solution sets of problems (QEP) by $S(\varphi, K)$ and $(\text{QEP})_n$ by $S(\varphi_n, K_n)$. Throughout this paper, we always assume that $S(\varphi, K)$, $S(\varphi_n, K_n)$ are not equal empty sets.

In the following, we introduce some concepts of the convergence of set sequences and mapping sequences (see [12]).

Let X be a normed space. A sequence of sets $\{D_n \subseteq X\}$ is said to *upper converge* (resp. *lower converge*) in the sense of Painlevé-Kuratowski to D if $\limsup\limits_{n\to\infty} D_n \subseteq D$ (resp. $D \subseteq \liminf\limits_{n\to\infty} D_n$). $\{D_n\}$ is said to *converge* in the sense of Painlevé-Kuratowski to D if $\limsup\limits_{n\to\infty} D_n \subseteq D \subseteq \liminf\limits_{n\to\infty} D_n$ with

$$\liminf_{n\to\infty} D_n := \{x \in X : x = \lim_{n\to\infty} x_n, x_n \in D_n \text{ for sufficiently large } n\},$$

$$\limsup_{n\to\infty} D_n := \{x \in X : x = \lim_{k\to\infty} x_{n_k}, x_{n_k} \in D_{n_k} \ \{n_k\} \text{ a subsequence of } \{n\}\}.$$

Let $G_n : X \to 2^Y$ be a sequence of set-valued mappings and $G : X \to 2^Y$ be a set-valued mapping. $\{G_n\}$ is said to *outer converge continuously* (resp. *inner converge continuously*) to G at x_0 if $\limsup\limits_{n\to\infty} G_n(x_n) \subseteq G(x_0)$ (resp. $G(x_0) \subseteq$

$\liminf_{n\to\infty} G_n(x_n))$ $\forall x_n \to x_0$. $\{G_n\}$ is said to *converge continuously* to G at x_0 if $\limsup\limits_{n\to\infty} G_n(x_n) \subseteq G(x_0) \subseteq \liminf\limits_{n\to\infty} G_n(x_n) \quad \forall x_n \to x_0$. If $\{G_n\}$ converges continuously to G at every $x_0 \in X$, then it is said that $\{G_n\}$ converges continuously to G on X.

The next, we introduce some concepts of the convergence of functional sequence (see [8]).

A sequence of functions $\{g_n\}$ where $g_n : X \to Y$ is said to *converge continuously* to a function $g : X \to Y$ at x_0 if $\lim\limits_{n\to\infty} g_n(x_n) = g(x_0) \quad \forall x_n \to x_0$. If $\{g_n\}$ converges continuously to g at every $x_0 \in X$, then it is said that $\{g_n\}$ converges continuously to g on X.

3 Main Results

In this section, our focus is on the Painlevé-Kuratowski upper convergence of the solution sets for $(\mathrm{QEP})_n$.

Theorem 3.1 *Suppose that*

(i) $K_n(.)$ *converges continuously to* $K(.)$ *and has compact values in* A*;*

(ii) $\varphi_n(.,.)$ *converges continuously to* $\varphi(.,.)$*.*

Then, $\limsup\limits_{n\to\infty} S(\varphi_n, K_n) \subseteq S(\varphi, K)$.

Proof. On a contrary, we suppose that $\limsup_{n\to\infty} S(\varphi_n, K_n) \nsubseteq S(\varphi, K)$, i.e, there exists $x_0 \in \limsup\limits_{n\to\infty} S(\varphi_n, K_n)$, but $x_0 \notin S(\varphi, K)$. Since $x_0 \in \limsup\limits_{n\to\infty} S(\varphi_n, K_n)$, there exists a sequence $\{x_m\} \subseteq S(\varphi_m, K_m)$, $x_m \to x_0$, $\forall m$. Then, for each $y \in K_m(x_m)$, we have

$$\varphi_m(x_m, y) \subseteq Y \setminus -\mathrm{int}C. \tag{3.1}$$

By $K_n(.)$ outer converges continuously to $K(.)$ and has compact values in A, with $x_m \in K_m(x_m)$ we have $x_0 \in K(x_0)$. As $x_0 \notin S(\varphi, K)$, there exists $y_0 \in K(x_0)$, such that

$$\varphi(x_0, y_0) \in -\mathrm{int}C. \tag{3.2}$$

By $K_n(.)$ inner converges continuously to $K(.)$ and $y_0 \in K(x_0)$, it is clear that, there exists $\{y_m\} \subseteq K_m(x_m)$ and $y_m \to y_0$. From (3.1), we have

$$\varphi_m(x_m, y_m) \subseteq Y \setminus -\mathrm{int}C. \tag{3.3}$$

As $\varphi_n(.,.)$ converges continuously to $\varphi(.,.)$ and by (3.3) with $(x_m, y_m) \to (x_0, y_0)$ and $Y \setminus -\text{int}C$ be closed, then

$$\varphi(x_0, y_0) \subseteq Y \setminus -\text{int}C. \tag{3.4}$$

We see a contradiction between (3.4) and (3.2) and so completed the proof. □

The following example shows that the condition (i) is essential.

Example 3.1 Let $X = Y = \mathbb{R}, C = \mathbb{R}_+, A = [-1, 1]$, consider the mappings $K : A \to 2^A, K_n : A \to 2^A$ such that $K(x) = [\frac{1}{2}, 1]$ and $K_n(x) = [-\frac{1}{n}, \frac{1}{n}]$. We define the mappings $\varphi : A \times A \to Y$ and $\varphi_n : A \times A \to Y$ such that

$$\varphi(x, y) = y - x \text{ and } \varphi_n(x, y) = (1 + \frac{1}{n})(y - x).$$

It follows from a direct computation $S(\varphi, K) = \{\frac{1}{2}\}$ and $S(\varphi_n, K_n) = \{-\frac{1}{n}\}$. We show that assumption (ii) of Theorem 3.1 is satisfied. But, $K_n(.)$ does not converge continuously to $K(.)$. Thus, the conclusion of Theorem 3.1 does not hold. In fact, $\{0\} = \limsup_{n\to\infty} S(\varphi_n, K_n) \nsubseteq S(\varphi, K) = \{\frac{1}{2}\}$.

The following example shows that the condition (ii) is essential.

Example 3.2 Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, A = [0, 1]$, consider the mappings $K : A \to 2^A, K_n : A \to 2^A$ such that $K(x) = K_n(x) = [0, 1]$. We define the mappings $\varphi : A \times A \to Y$ and $\varphi_n : A \times A \to Y$ such that

$$\varphi(x, y) = (-1 - x, \frac{1}{2} - x) \text{ and } \varphi_n(x, y) = (-1 - x, -\frac{1}{n} + \frac{x}{n}).$$

It follows from a direct computation $S(\varphi, K) = [0, \frac{1}{2}]$ and $S(\varphi_n, K_n) = \{1\}$. We show that assumption (i) of Theorem 3.1 is satisfied. But, $\varphi_n(.,.)$ does not converge continuously to $\varphi(.,.)$. Thus, the conclusion of Theorem 3.1 does not hold. In fact, $\{1\} = \limsup_{n\to\infty} S(\varphi_n, K_n) \nsubseteq S(\varphi, K) = [0, \frac{1}{2}]$.

The next example shows that all assumptions of Theorem 3.1 are fulfilled.

Example 3.3 Let $X = Y = \mathbb{R}, C = \mathbb{R}_+, A = [-4, 4]$, consider the mappings $K : A \to 2^A, K_n : A \to 2^A$ such that $K(x) = [-1, 3]$ and $K_n(x) = [-1 - \frac{1}{n}, 3 + \frac{1}{n}]$. We define the mappings $\varphi : A \times A \to Y$ and $\varphi_n : A \times A \to Y$ such that

$$\varphi(x, y) = 3 - 2x - x^2 \text{ and } \varphi_n(x, y) = 3 - 2(x + \frac{1}{n}) - (x + \frac{1}{n})^2.$$

It follows from a direct computation $S(\varphi, K) = [-1, 1]$ and $S(\varphi_n, K_n) = [-1 - \frac{1}{n}, 1 - \frac{1}{n}]$. We show that assumptions (i) and (ii) of Theorem 3.1 are satisfied. Then, the result of Theorem 3.1 hold. In fact, $\limsup_{n\to\infty} S(\varphi_n, K_n) = [-1, 1] = S(\varphi, K)$.

Remark 3.1

Let X and Y be two normed linear spaces and $C = E$, E is improvement set (see Definition 2.1 in [9]), $K(x) = S, K_n(x) = S_n, \forall x \in A$. Let $\varphi(x,y) = f(y) - f(x), \varphi_n(x,y) = f_n(y) - f_n(x)$ (with f, f_n be functions from X into Y) for any $x, y \in X$. Then, the $(\text{QEP})_n$ reduce to a sequence of vector optimization problem (in short, $(\text{P})_n$) and the (QEP) reduced to the vector optimization problem (in short, (P)), this problem is studied in [9]. Then, Theorem 3.3 and Theorem 4.6 (in [9]) are special cases of Theorem 3.1. Moreover, the proof of Theorem 3.1 is different from Theorem 3.3 and Theorem 4.6 in [9].

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