ON RADICALS OF LEFT V-SEMIRINGS

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Abstract. I*n this paper, we solve Problem 1 in [8] for left V-semirings. Specifically, we prove that has the inclusion of the radical $J_s(R)$ into the Jacobson radical $J(R)$ for every left V-semiring R. Moreover, we give a necessary and sufficient condition for two radicals are equal on left artinian (or subtractive) left V-semirings.*

1 Introduction

Hemirings, introduced by Vandiver [13] in 1934, generalize the notion of noncommutative rings in the sense that negative elements do not have to exist. Since then there have been an active area of research in hemirings, both on the theoretical side and on the side of applications e.g. in theoretical computer science. The reader may consult the monographs of Golan [4] for a more elaborate introduction to hemirings.

The Jacobson radical $J(R)$ of a hemiring R was first introduced by S. Bourne in an internal way (see, [2]). The Jacobson radical of a hemiring has subsequently been studied by Iizuka from the point of view of representation theory (see, [6]). However, if R is an additively idempotent hemiring then the Jacobson radical $J(R) = R$ (see, [12, Proposition 2.5] or [8, Example 3.7]). Thus, for additively idempotent hemirings, Jacobson radical is not a good tool. Recently, Katsov and Nam have defined radical $J_s(R)$ of a hemiring R (see, [8, p. 5076]). And they also confirmed that $J_s(R) \subseteq J(R)$ for any commutative or additively regular, in particular additively idempotent, hemiring R [8, Proposition 4.8]. And raised a problem as follow: Describe the subclass of all hemirings R of the class $\mathbb{H}$ ($\mathbb{H}$ be the set of all hemirings) with $J_s(R) \subseteq J(R)$, particular, with $J_s(R) = J(R)$ [8, Problem 1]. In this paper we will solve Problem 1 in [8] for left V-semiring class. The paper is organized as follows.

In Section 2, for the reader's convenience, we included all subsequently necessary notions and facts on semirings and semimodules, as well as on the radical theory of semirings. In Section 3, we prove a radical operator distributed for a direct sum of semirings (Proposition 3.3), then inferred for the radical operators J and J_s (Corollary 3.4). Next, we resolve Problem 1 in [8], with the inclusion between the above two radicals for left V-semirings (Theorem 3.5). We prove that, if R is a left artinian (or subtractive) semiring then there exists simple left R-semimodule (Corollary 3.7), from this corollary we confirmed $J_s(R) \subsetneq J(R) = R$ for a left artinian (or subtractive) zeroic semiring (Corollary 3.8). We conclude this section with an answer when two radicals are equal for left artinian (or subtractive) left V-semirings (Theorem 3.9).

Finally, all notions and facts of radical theory of semirings, use here without any comments, can be found in [2, 6, 8, 11]; for notions and facts from semirings and semimodules we refer to [4].

2 Preliminaries

Recall from [4] that a *hemiring* R is an algebra $(R, +, \cdot, 0)$ such that the following conditions are satisfied:

(1) $(R, +, 0)$ is a commutative monoid with identity element 0;

(2) $(R, \cdot)$ is a semigroup;

(3) Multiplication distributes over addition on either side;

(4) $0r = 0 = r0$ for all $r \in R$.

A hemiring R is called a *semiring* if its multiplicative semigroup $(R, \cdot)$ is a monoid with identity element 1. A hemiring R is called *additively cancellative* if $a+c = b+c$ implies $a = b$ for all $a, b, c \in R$. The notions of an *(two-sided) ideal*, a *left ideal* and a *right ideal* of a hemiring R are defined similarly as for rings. The *subtractive closure* $\overline{I} = \{r \in R \mid r + i \in I \text{ for some } i \in I\}$ of an ideal I is an ideal of R. An ideal I of a hemiring R is called *subtractive* if $\overline{I} = I$; that is, for all $x, a \in R$, if $x + a, a \in I$ then $x \in I$. Denote by $\mathcal{I}(R)$ and $\mathcal{SI}(R)$ the sets of all ideals and all subtractive ideals of R, respectively. A semiring is called *right (left) subtractive* if every right (left) ideal is subtractive. The subset $Z(R) = \{r \in R \mid r + x = x \text{ for some } x \in R\}$ denote the *zeroic* of a hemiring R. A hemiring R is *zeroic* if $Z(R) = R$.

As for rings, for any homomorphism $f : R \to S$ between hemirings R and S, there exists a subtractive ideal, the *kernel*, $Kerf = \{r \in R \mid f(r) = 0\} \subseteq R$ of f. A surjective hemiring homomorphism $f : R \to S$ is a *semiisomorphism* if $Kerf = 0$. As usual, the direct product $R = \prod_{i\in I} R_i$ of a family $(R_i)_{i\in I}$ of hemirings R_i consists of the elements $r = (r_i)_{i\in I}$ for $r_i \in R_i$ and is determined by the surjective homomorphisms $\pi_i : R \to R_i$ defined by $\pi_i(r) = r_i$; and a subhemiring S of R is called a *subdirect product* $S = \prod_{i\in I}^{sub} R_i$ of $(R_i)_{i\in I}$ if, for each π_i, the restriction $\pi_i|_S : S \to R_i$ is also surjective.

Any ideal I of a hemiring R induces on R a congruence relation $\equiv_I$, which is referred to as *Bourne relation* [4, p.78] and is given by: $r \equiv_I r'$ iff there exist elements $i_1, i_2 \in I$ such that $r + i_1 = r' + i_2$. Denote the factor hemiring $R/{\equiv_I}$ by R/I. It is easy to see that $\equiv_I$ and $\equiv_{\overline{I}}$ on R coincide for every ideal I of R, and hence $R/I = R/\overline{I}$ holds for every ideal I of R.

As usual, a *left R-semimodule* over a hemiring R is a commutative monoid $(M, +, 0_M)$ together with a scalar multiplication $(r, m) \mapsto rm$ from $R \times M$ to M that satisfies the following identities for all $r, r' \in R$ and $m, m' \in M$:

(1) $(rr')m = r(r'm)$;

(2) $r(m + m') = rm + rm'$;

(3) $(r + r')m = rm + r'm$;

(4) $r0_M = 0_M = 0m$.

Right R-semimodules and homomorphisms between semimodules are defined in the standard manner. If a hemiring R is a semiring, then all semimodules over R are unitary ones. Denote by $\mathcal{M}_R$ and ${}_R\mathcal{M}$ the categories of all right and left R-semimodules, respectively. A semimodule M is *injective* if each R-homomorphism $\varphi : A \to M$ may be extended to an R-homomorphism $\overline{\varphi} : B \to M$ for every R-semimodule B and every subsemimodule $A \subseteq B$. A left semimodule M over a hemiring R is *cancellative* if $x + z = y + z$ implies $x = y$ for all $x, y, z \in M$. A subsemimodule N of an R-semimodule M is *subtractive* if, for all $x, y \in M$, from

$x + y, x \in N$ it follows that $y \in N$, too.

The usual concepts of the *Descending Chain Condition* and *artinian* modules of theory of modules over rings, as well as results involving them, are easily extended in an obvious fashion (see, for example, [7]) to a context of semimodules over semirings. For a left R-semimodule ${}_RM$, the ideal $(0 : M)_R = \{r \in R \mid rM = 0\}$ of R is called the *annihilator* of M.

Congruences on an R-semimodule M are defined in the standard manner, and $\mathrm{Cong}(M)$ denotes the set of all congruences on M. This set is non-empty since it always contains at least two trivial congruences, the *diagonal congruence* $\Delta_M := \{(m, m) \mid m \in M\}$ and the *universal congruence* $M^2 := \{(m, n) \mid m, n \in M\}$. A semimodule $M \neq 0$ is *congruence-simple* provided that $\mathrm{Cong}(M) = \{\Delta_M, M^2\}$. Any subsemimodule N of an R-semimodule M induces a congruence $\equiv_N$ on M, known as the *Bourne congruence*, by setting $m \equiv_N m'$ iff $m + n = m' + n'$ for some $n, n' \in N$; and M/N denotes the factor R-semimodule $M/\equiv_N$, having the canonical R-surjection $\pi_N : M \to M/N$.

A nonzero cancellative left semimodule M over a hemiring R is *irreducible* if, for an arbitrarily fixed pair of elements $m_1, m_2 \in M$ with $m_1 \neq m_2$ and any $m \in M$, there exist $r_1, r_2 \in R$ such that $m + r_1m_1 + r_2m_2 = r_1m_2 + r_2m_1$. By [6, Definition 6], the *Jacobson radical*

$$J(R) = \cap\{(0 : M)_R \mid M \in \mathcal{J}\},$$

for $\mathcal{J}$ be the set of all irreducible left semimodules over a hemiring R. When $\mathcal{J} = \emptyset$, by convention, $J(R) = R$.

A left R-semimodule M is *congruence-simple* if $\mathrm{Cong}(M) = \{\Delta_M, M^2\}$. A left R-semimodule M is *simple* if $RM \neq 0$ and there are only trivial subsemimodules of, as well as congruences on, M. Call a radical

$$J_s(R) = \cap\{(0 : M)_R \mid M \in \mathcal{J}'\},$$

for $\mathcal{J}'$ be the set of all simple left semimodules over a hemiring R. When $\mathcal{J}' = \emptyset$, by convention, $J_s(R) = R$ (see, [8, p. 5076]).

Remark If M is a simple left semimodule over a semiring R then M is always unitary, that is, $1.m = m$ for all $m \in M$. Indeed, $r(1.m) = (r.1)m = rm$ for all $r \in R$ and consider the congruence ρ on M given by: $x\rho y$ if and only if $rx = ry$ for all $r \in R$ and $x, y \in M$. Since M is simple, $\rho = \Delta_M$ or $\rho = M^2$. If $\rho = M^2$, then $(x, 0) \in \rho$ for all $x \in M$, that is, $rx = 0$ for all $r \in R$ and $x \in M$, and hence, $RM = 0$ (contraction). Thus, $\rho = \Delta_M$, that is, $(1.m, m) \in \rho$, i.e., $1.m = m$.

In the Kurosh-Amitsur radical theory of the category $\mathbb{H}$ of all hemirings [11, p. 536], a nonempty subclass $\mathbb{U}$ of $\mathbb{H}$ is said to be *hereditary* if $R \in \mathbb{U}$ implies $\mathcal{I}(R) \subseteq \mathbb{U}$, and *homomorphically closed* if $R \in \mathbb{U}$ implies $\varphi(R) \in \mathbb{U}$ for each homomorphism φ of R. If $\mathbb{U}$ is both hereditary and homomorphically closed, then it is said to be *universal*. As in the radical theory of rings, there are three equivalent approaches to the Kurosh-Amitsur radical theory of hemirings, by means of radical classes, of radical operators, and of semisimple classes. These approaches are independently defined in a fixed universal class $\mathbb{U} \subseteq \mathbb{H}$ of hemirings. Define $\mathbb{T} = \{S \in \mathbb{H} \mid |S| = 1\}$ as the class of all trivial hemirings.

A nonempty subclass $\mathbb{R}$ of a fixed universal class $\mathbb{U} \subseteq \mathbb{H}$ is called a *radical class* of $\mathbb{U}$ if $\mathbb{R}$ satisfies the following two conditions [11, Defination 3.1]:

(1) $\mathbb{R}$ is homomorphically closed;

(2) For every hemiring $R \in \mathbb{U} \setminus \mathbb{R}$, there is a subtractive ideal $K \in \mathcal{SI}(R) \setminus \{R\}$ such that $\mathcal{I}(R/K) \cap \mathbb{R} = 0$.

A mapping $\rho : \mathbb{U} \to \mathbb{U}$ is called a *radical operator* in $\mathbb{U}$ if it assigns to each hemiring $R \in \mathbb{U}$ a subtractive ideal $\rho(R) \in \mathcal{SI}(R) \subseteq \mathbb{U}$ such that the following conditions are satisfied for all $S, T \in \mathbb{U}$ [11, Defination 4.1]:

(1) $\varphi(\rho(S)) \subseteq \rho(\varphi(S))$ for each homomorphism $\varphi : S \to T$;

(2) $\rho(S/\rho(S)) = 0$;

(3) if $\rho(T) = T$ is an ideal of S then $T \subseteq \rho(S)$;

(4) $\rho(\rho(S)) = \rho(S)$.

3 Main results

A concept of left V-ring was defined by O. Villamajor in the rings and modules (see, for example, [9]). Generalizing the well known for rings notions and following [5], we call a semiring R a *left V-semiring* if every congruence-simple left R-semimodule is injective; and an R-semimodule M is called an *essential extension* of an R-subsemimodule L, $i : L \rightarrowtail M$, if for every semimoddule homomorhpism $\gamma : M \to N$, the homomorphisms γi and γ are simultaneously injective.

Example 3.1. *(i) A zeroic proper division semiring or a division ring is a left V-semiring according to [1, Corollary 3.5].*

(ii) [1, Example 3.7] Let n be a nonzero natural number and $\mathbb{B}_{n+1}$ the join-semilattice defined on the chain $0 < 1 < ... < n$. Equip $\mathbb{B}_{n+1}$ with a structure of a semiring with addition $x + y := x \vee y$ and multiplication $xy := 0$ if $x = 0$ or $y = 0$ and $xy := x \vee y$ otherwise. Then $\mathbb{B}_{n+1}$ is a left V-semiring. Of course, $\mathbb{B}_2$ coincides with Boolean semiring $\mathbb{B}$.

(iii) Finite Boolean algebra $D = \mathbb{B} \times \mathbb{B} \times ... \times \mathbb{B}$ is a left V-semiring according to [1, Corollary 3.2].

Now, we recall the main result of [5], which is used in main section of this paper.

Theorem 3.2 (5, Theorem 2.10)**.** *For a semiring R the following are equivalent:*

(1) R is a left V-semiring;

(2) Every essential extension of each congruence-simple left R-semimodule M coincides with M;

(3) $R = S \oplus T$, where S is a left V-ring and T is a zeroic left V-semiring;

(4) Each quotient semiring of R is a left V-semiring.

First, we prove a result concerning a radical operator. Specifically, the radical operator distributes for the direct sum of the hemirings.

Proposition 3.3. *Let $\mathbb{R}$ be a radical class in a universal class $\mathbb{U}$ and ρ be a radical operator corresponding. Then $\rho(A \oplus B) = \rho(A) \oplus \rho(B)$ for every $A, B \in \mathbb{U}$.*

By [2, Theorems 5 and 6], the mapping $\varrho : \mathbb{H} \to \mathbb{H}$ given by $R \mapsto J(R)$ is, in fact, a radical operator in $\mathbb{H}$. The same has been proved in [8] that the mapping $\varrho : \mathbb{H} \to \mathbb{H}$ given by $R \mapsto J_s(R)$ is also a radical operator in $\mathbb{H}$. From these observations and using Proposition 3.3, we have the following.

Corollary 3.4. *Let R be a hemiring and R_1, R_2 its subhemirings. If $R = R_1 \oplus R_2$ then $J(R) = J(R_1) \oplus J(R_2)$; and $J_s(R) = J_s(R_1) \oplus J_s(R_2)$.*

Next, we solve Problem 1 in [8], the answered inclusion between the two radicals, over left V-semirings.

Theorem 3.5. *If R is a left V-semiring, then $J_s(R) \subseteq J(R)$.*

Following, we prove that always build a simple left R-semimodule from a minimal left R-semimodule.

Lemma 3.6. *Let M be a minimal left R-semimodule. Then there exists a maximal congruence ρ on M such that $\overline{M} := M/\rho$ is a simple left R-semimodule.*

According to [7, Proposition 2.1], M is left artinian (subtractive) left R-semimodule if and only if every non-empty set of subsemimodules of M has a minimal element. From this result and Lemma 3.6, we have the following.

Corollary 3.7. *If R is a left artinian (or subtractive) semiring then there exists a simple left R-semimodule.*

Corollary 3.8. *For a left artinian (or subtractive) zeroic semiring R, $J_s(R) \subsetneq J(R) = R$.*

Now, we solve Problem 1 in [8], answered when two radicals are equal, over left artinian (or subtractive) left V-semirings.

Theorem 3.9. *For a left artinian (or subtractive) left V-semiring R, $J_s(R) = J(R)$ if and only if R is a left V-ring.*

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